

Lifting and Projection

Roots, Rotations and the Rank of a State

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ABSTRACT We study a single elementary mechanism in several guises: a problem posed in a space of dimension n whose solution, or whose simplest description, lives in a space of larger dimension, and the loss of information incurred when that solution is projected back. First, we prove a trichotomy for real matrices with $M^3 = \varepsilon M$, $\varepsilon \in \{-1, 0, 1\}$, whose exponentials give rotations (Rodrigues), shears (translations and Galilean boosts) and hyperbolic rotations (Lorentz boosts) by one formula. Second, we determine the solutions of a real quadratic equation with negative discriminant in several real algebras: none in $\mathbb{R}$, two in $\mathbb{C}$, none in $M_n(\mathbb{R})$ for odd n , a two-sheeted hyperboloid in $M_2(\mathbb{R})$, and a two-sphere in the quaternions; the two complex roots are the trace of the sphere on a complex slice. Third, we record three artefacts of orthogonal projection, namely apparent annihilation, orientation reversal and non-identifiability, and the condition under which several projections recover the state. Fourth, we realise the Galilean group in $GL(n+2, \mathbb{R})$, so that velocity is a shear fed by the time coordinate, and compute an explicit example from robot kinematics. Fifth, we express the separability of a composite state as a rank condition, give a twelve-component real model in which a single nilpotent matrix fixes two separated outcomes in one step, and recall why models built from two separate objects satisfy the Bell–CHSH bound while the single-object model does not.

KEYWORDS Rodrigues’ formula; nilpotent and hyperbolic generators; quadratic equations in real algebras; quaternions; orthogonal projection; homogeneous coordinates; Galilean group; Schmidt rank; Bell–CHSH inequality

I. INTRODUCTION

Much of elementary mathematics is the art of choosing the space in which a problem is simple. The equation $x^2 + 4 = 0$ has no solution on the line and two in the plane; perspective is non-linear in three coordinates and linear in four; the harmonic oscillator is second order in one variable and a first-order rotation in two. In each case a problem is *lifted* to a larger space, solved there, and *projected* back.

The present paper isolates this mechanism and studies it with complete proofs in a number of concrete settings. None of the individual results is deep, and several are classical; the interest lies in placing them side by side, in making precise the sense in which “the number of solutions” or “the number of objects” depends on the ambient space, and in separating what a projection preserves from what it destroys. Throughout, all vector spaces are real and finite-dimensional unless stated otherwise, E_{pq} denotes the matrix unit with a 1 in row p and column q , and $\{e_k\}$ is the standard basis. Numerical values quoted were computed in IEEE double precision; exact values are given where they are known.

2. THREE KINDS OF GENERATOR

Definition 2.1. A matrix $M \in M_n(\mathbb{R})$ is a *generator of type* $\varepsilon \in \{-1, 0, 1\}$ if $M^3 = \varepsilon M$.

The three types are realised by the elementary matrices that underlie Euclidean, affine and Lorentzian geometry. For $p \neq q$ put

$$M_{pq} = E_{qp} - E_{pq}, \quad N_{pq} = E_{pq}, \quad K_{pq} = E_{pq} + E_{qp}. \quad (2.1)$$

A direct computation gives $M_{pq}^2 = -(E_{pp} + E_{qq})$, hence $M_{pq}^3 = -M_{pq}$; $N_{pq}^2 = 0$; and $K_{pq}^2 = E_{pp} + E_{qq}$, hence $K_{pq}^3 = K_{pq}$. The matrix M_{pq} turns e_p towards e_q in the plane they span, N_{pq} feeds the q -th coordinate into the p -th, and K_{pq} exchanges them symmetrically.

Lemma 2.2. If $M^3 = \varepsilon M$ then $M^{2k+1} = \varepsilon^k M$ and $M^{2k+2} = \varepsilon^k M^2$ for all $k \geq 0$, with the convention $0^0 = 1$.

Proof. Both identities hold for $k = 0$. If they hold for k , then $M^{2k+3} = M^{2k+1} M^2 = \varepsilon^k M^3 = \varepsilon^{k+1} M$ and $M^{2k+4} = M^{2k+2} M^2 = \varepsilon^k M^4 = \varepsilon^{k+1} M^2$. $\square$

Theorem 2.3 (Rodrigues' formula and its two companions). *Let M be a generator of type ε . For every $\theta \in \mathbb{R}$,*

$$e^{\theta M} = I + S_\varepsilon(\theta) M + C_\varepsilon(\theta) M^2, \quad (2.2)$$

where $S_{-1} = \sin$, $C_{-1} = 1 - \cos$; $S_0(\theta) = \theta$, $C_0(\theta) = \theta^2/2$; and $S_1 = \sinh$, $C_1 = \cosh - 1$.

Proof. The exponential series converges absolutely, so it may be split by the parity of the exponent. By Lemma 2.2,

$$e^{\theta M} = I + \left(\sum_{k \geq 0} \frac{\varepsilon^k \theta^{2k+1}}{(2k+1)!} \right) M + \left(\sum_{k \geq 0} \frac{\varepsilon^k \theta^{2k+2}}{(2k+2)!} \right) M^2.$$

For $\varepsilon = -1$ the two series are those of $\sin \theta$ and $1 - \cos \theta$; for $\varepsilon = 1$ those of $\sinh \theta$ and $\cosh \theta - 1$; for $\varepsilon = 0$ only the terms with $k = 0$ survive. $\square$

For $\varepsilon = -1$ this is the rotation formula usually attributed to Rodrigues (Rodrigues 1840). On the plane of M_{pq} , where $M^2 = -I$, it reduces to $\cos \theta I + \sin \theta M$, which is Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ (Euler 1748) with the real matrix M in place of i . For $N = N_{pq}$ one has $M^2 = 0$ as well, and (2.2) collapses to $e^{dN} = I + dN$: a shear. For $K = K_{pq}$ it gives $e^{\varphi K} = I + \sinh \varphi K + (\cosh \varphi - 1)K^2$, a hyperbolic rotation of the plane (e_p, e_q) that fixes the other coordinates.

Remark 2.4. The three cases are related by contraction. In the (x, t) plane, with the speed of light c restored, the Lorentz boost of velocity v is $\Lambda_c = \gamma \begin{pmatrix} 1 & v/c^2 \\ v/c & 1 \end{pmatrix}$, $\gamma = (1 - v^2/c^2)^{-1/2}$, and $\Lambda_c \rightarrow \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = e^{vN}$ as $c \rightarrow \infty$. This is the simplest instance of the group contractions of İnönü and Wigner (İnönü and Wigner 1953).

3. A QUADRATIC EQUATION IN SEVERAL ALGEBRAS

Let $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, $a \neq 0$, discriminant $\Delta = b^2 - 4ac$ and vertex $b = -b/2a$. For any real associative unital algebra A we ask for the solutions of $f(x) = 0$ in A , reading f with real coefficients.

Lemma 3.1 (the lifted parabola). *Let $u \in A$ satisfy $u^2 = -1$. Then for every $t \in \mathbb{R}$,*

$$f(b + tu) = f(b) - at^2 = -\frac{\Delta}{4a} - at^2. \quad (3.1)$$

Proof. Since b and t are real they commute with u , and $(b + tu)^2 = b^2 + 2btu - t^2$. Hence $f(b + tu) = f(b) - at^2 + t(2ab + b)u$, and $2ab + b = 0$. $\square$

Geometrically, Lemma 3.1 says that above the vertex, in the direction u , the graph of f is the parabola $f(b) - at^2$, which opens in the opposite direction to the original. If $\Delta < 0$ the original parabola does not meet the axis, and the lifted one does, at $t = \pm\sqrt{-\Delta}/2a$. For $f(x) = x^2 + 4$ the lifted values at $t = 0, 1, 2, 3$ are $4, 3, 0, -5$.

Theorem 3.2. *Suppose $\Delta < 0$. For every $u \in A$ with $u^2 = -1$ the elements*

$$x_{\pm} = -\frac{b}{2a} \pm \frac{\sqrt{-\Delta}}{2a} u \quad (3.2)$$

are solutions of $f(x) = 0$ in A .

Proof. Immediate from Lemma 3.1 with $t = \pm\sqrt{-\Delta}/2a$. $\square$

The set of solutions therefore depends on the set of square roots of -1 in A , and on whether further solutions exist. We treat the cases in turn.

Proposition 3.3 (the line and the plane). *In $A = \mathbb{R}$ there is no solution. In $A = \mathbb{C}$ there are exactly two, $b \pm \frac{\sqrt{-\Delta}}{2a} i$. Identifying $\mathbb{C}$ with the subalgebra $\{\alpha I + \beta J\} \subset M_2(\mathbb{R})$, $J = M_{01}$, these are the real matrices $bI \pm \frac{\sqrt{-\Delta}}{2a} J$.*

Proof. The first statement is the sign of f on $\mathbb{R}$. In the field $\mathbb{C}$ a quadratic has at most two roots, and Theorem 3.2 provides two. The identification is the standard one, $J^2 = -I$. $\square$

For $f(x) = x^2 - 2x + 5$ the root is $X = I + 2J = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, and $X^2 - 2X + 5I$ vanishes identically. Nothing in the computation involves a non-real number.

Proposition 3.4 (odd dimension). *If n is odd and $\Delta < 0$, then $f(X) = 0$ has no solution $X \in M_n(\mathbb{R})$. In particular no real $n \times n$ matrix squares to $-I$.*

Proof. If $f(X) = 0$, every eigenvalue of X is a root of f , so X has no real eigenvalue. But the characteristic polynomial of X has odd degree and real coefficients, and therefore a real root. $\square$

Three dimensions nevertheless contain the solutions of the plane, one for each axis. For a unit vector $k \in \mathbb{R}^3$ let K_k be the matrix of $v \mapsto k \times v$.

Proposition 3.5 (Rodrigues' generator). *$K_k^3 = -K_k$ and $K_k^2 = kk^\top - I$. Consequently $(2K_k)^2 + 4I = 4kk^\top$: the matrix $2K_k$ solves $X^2 + 4I = 0$ on the invariant plane $k^\perp$ and on no larger subspace.*

Proof. By the vector triple product, $K_k^2 v = k \times (k \times v) = (k \cdot v)k - v$, which is $(kk^\top - I)v$. Then $K_k^3 = K_k(kk^\top - I) = -K_k$ because $K_k k = 0$. The last assertion follows because $4kk^\top$ vanishes exactly on $k^\perp$. $\square$

For $k = (1, 2, 2)/3$ the residual $(2K_k)^2 + 4I$ annihilates $(2, -1, 0)$ and $(0, 1, -1)$ and sends k to $4k$; the quarter-turn $R = e^{(\pi/2)K_k}$ satisfies $R^2 = 2kk^\top - I$, which is $-I$ on $k^\perp$ and $+I$ on the axis. Proposition 3.4 explains why the axis must survive.

Proposition 3.6 (the matrices of order two). *The set $\{X \in M_2(\mathbb{R}) : X^2 = -I\}$ is*

$$\left\{ \begin{pmatrix} p & q \\ r & -p \end{pmatrix} : p^2 + qr = -1 \right\},$$

a hyperboloid of two sheets, the two sheets being the conjugacy classes of J and $-J$.

Proof. By the Cayley–Hamilton theorem $X^2 - (\text{tr } X)X + (\det X)I = 0$. If $X^2 = -I$ then X is not a real multiple of I , so comparing the two relations gives $\text{tr } X = 0$ and $\det X = 1$, that is, X has the stated form with $-p^2 - qr = 1$. With $q = s + w$, $r = s - w$ the condition reads $w^2 - p^2 - s^2 = 1$. The sign of $w = (q - r)/2$ separates the sheets. The quadratic form $v \mapsto \det(v, Xv) = rv_1^2 - 2pv_1v_2 - qv_2^2$ has discriminant $4(p^2 + qr) = -4$, so it is definite, and its sign, that of its trace $r - q = -2w$, is the orientation in which X turns the plane. $\square$

By Theorem 3.2 each point of this hyperboloid yields a root of f in $M_2(\mathbb{R})$. The two roots $\pm J$ of the commutative subalgebra $\mathbb{C}$ are the points $p = s = 0$.

Theorem 3.7 (quaternions). *Let $b, c \in \mathbb{R}$ and consider $x^2 + bx + c = 0$ in Hamilton’s quaternions $\mathbb{H}$ (Hamilton 1844).*

(i) *If $\Delta \geq 0$ the solutions are the real roots only.*

(ii) *If $\Delta < 0$ the solutions are exactly $x = -\frac{b}{2} + \frac{\sqrt{-\Delta}}{2} u$, where u ranges over the unit sphere S^2 of pure quaternions.*

Proof. Write $x = s + v$ with $s \in \mathbb{R}$ and v pure. Then $v^2 = -|v|^2$ and $x^2 = s^2 - |v|^2 + 2sv$. The equation splits into a real part and a pure part:

$$s^2 - |v|^2 + bs + c = 0, \quad (2s + b)v = 0.$$

If $v = 0$, $x = s$ is a real root. If $v \neq 0$, then $s = -b/2$ and $|v|^2 = c - b^2/4 = -\Delta/4$, which is possible only if $\Delta < 0$ (the case $\Delta = 0$ forces $v = 0$). Conversely every x of the stated form satisfies both equations. $\square$

Theorem 3.7 is a special case of Niven’s analysis of polynomial equations over the quaternions (Niven 1941); see also Eilenberg and Niven 1944 for the existence theorem.

Corollary 3.8. *For $\Delta < 0$ the quaternionic solution set is a round two-sphere of radius $\sqrt{-\Delta}/2$ centred at $-b/2$. Its intersection with any complex slice $\mathbb{C}_u = \text{span}\{1, u\}$, $u \in S^2$, consists of exactly two points, the complex roots.*

The count of solutions is thus a property of the pair (equation, algebra), not of the equation alone: for $x^2 + 4 = 0$ it is 0 in $\mathbb{R}$, 2 in $\mathbb{C}$, 0 in $M_3(\mathbb{R})$, a two-sheeted hyperboloid in $M_2(\mathbb{R})$, and a sphere in $\mathbb{H}$. The fundamental theorem of algebra (Gauss 1799) is a theorem about $\mathbb{C}$.

A real model of the quaternions. Left multiplication $L: \mathbb{H} \rightarrow M_4(\mathbb{R})$ in the basis $(1, i, j, k)$ is an injective homomorphism of real algebras, with

$$L(i) = M_{01} + M_{23}, \quad L(j) = M_{02} + M_{31}, \quad L(k) = M_{03} + M_{12}.$$

One checks $L(i)L(j)L(k) = -I$, Hamilton’s relation. For a unit pure quaternion $u = u_1i + u_2j + u_3k$, $L(u)^2 = -I$ on the whole of $\mathbb{R}^4$, unlike the three-dimensional case of Proposition 3.5,

and Theorem 2.3 with $M^2 = -I$ gives $e^{\theta L(u)} = \cos \theta I + \sin \theta L(u)$. The root of $x^2 + 4 = 0$ attached to u is the real matrix $2L(u) = 2e^{(\pi/2)L(u)}$: a quarter turn of $\mathbb{R}^4$ in two orthogonal planes at once, scaled by two. In floating point the residual $X^2 + 4I$ is exactly zero for $u \in \{(1, 0, 0), (0, 1, 0), (0, 0.6, 0.8), (1, 2, 2)/3\}$, and the residual of $X = I + 2L(u)$ in $x^2 - 2x + 5 = 0$ is at most 4.4×10^{-16} .

4. WHAT A PROJECTION DISCARDS

Let $V \subset \mathbb{R}^n$ be a subspace of dimension m and P the orthogonal projection onto V . We regard P as an observation that records V -components and discards the $(n - m)$ -dimensional kernel.

Proposition 4.1 (three artefacts). *Let $e_1 \in V$ and $e_n \perp V$, and let $R_\theta = e^{\theta M_{1n}}$.*

- (i) Apparent annihilation. $\|R_\theta e_1\| = 1$ for all θ , while $PR_\theta e_1 = \cos \theta e_1$. The observed vector shrinks, vanishes at $\theta = \pi/2$ and reappears reversed.
- (ii) Orientation reversal. The compression $C_\theta = PR_\theta|_V$ satisfies $\det C_\theta = \cos \theta$. Hence C_π is a reflection of V , although R_π lies in the identity component of $\text{SO}(n)$; the path $\theta \mapsto C_\theta$ from I to a reflection passes through the singular map $C_{\pi/2}$.
- (iii) Non-identifiability. States s, s' with $s - s' \in \ker P$ are observed identically.

Proof. R_θ acts as the identity off the plane (e_1, e_n) and sends $e_1 \mapsto \cos \theta e_1 + \sin \theta e_n$ and $e_n \mapsto -\sin \theta e_1 + \cos \theta e_n$. Projecting gives (i), and in the basis of V containing e_1 the compression is $\text{diag}(\cos \theta, 1, \dots, 1)$, which gives (ii). (iii) is linearity. $\square$

Part (ii) is the algebraic form of Möbius' observation that a rotation through a fourth dimension carries a three-dimensional figure onto its mirror image (Möbius 1827). In the case $n = 4$, $m = 3$ the compression at $\theta = \pi$ is $\text{diag}(-1, 1, 1)$, and the two states $(1, 2, 3, 0)$ and $(1, 2, 3, 7)$, at distance 7, have the same image.

Proposition 4.2 (recovery from several views). *Let $P_1, \dots, P_r$ be linear maps on $\mathbb{R}^n$. The stacked map $s \mapsto (P_1 s, \dots, P_r s)$ is injective if and only if $\bigcap_i \ker P_i = \{0\}$.*

Proof. The kernel of the stacked map is the intersection of the kernels. $\square$

A ball moving in a vertical plane, $(x, y)(t)$, casts under a sun of elevation α the shadow $s = x + y \cot \alpha$, a single row $P_\alpha = (1, \cot \alpha)$. From s alone the apparent acceleration is $g \cot \alpha$ and the horizontal and vertical speeds are confounded. Two elevations $\alpha_1 \neq \alpha_2$ give an invertible 2×2 system, from which the hidden height $y(t)$ is recovered exactly.

5. HOMOGENEOUS COORDINATES AND VELOCITY

5.1 Position as a shear

Write a point of $\mathbb{R}^n$ as (x, w) with $w = 1$. A translation by d is the shear $e^{\sum_i d_i N_{i,w}} = I + \sum_i d_i N_{i,w}$, and the affine group embeds in $\text{GL}(n+1, \mathbb{R})$ as the matrices $\begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}$ (Möbius 1827). The kinematic chains of robotics are products of such matrices (Denavit and Hartenberg 1955); rotations about joints are of type -1 and link offsets of type 0 in the sense of Section 2.

5.2 Velocity as a shear

Add a time coordinate and write an event as $(x, w, t) \in \mathbb{R}^{n+2}$ with $w = 1$.

Proposition 5.1. *The map*

$$(R, a, v, \tau) \mapsto G(R, a, v, \tau) = \begin{pmatrix} R & a & v \\ 0 & 1 & 0 \\ 0 & \tau & 1 \end{pmatrix}, \quad R \in \text{SO}(n), \ a, v \in \mathbb{R}^n, \ \tau \in \mathbb{R},$$

is an injective homomorphism from the Galilean group into $\text{GL}(n+2, \mathbb{R})$. It acts on events by $x \mapsto Rx + a + vt$, $t \mapsto t + \tau$, and composition obeys $v = v_1 + R_1 v_2$.

Proof. Multiply two such matrices: the upper-left block is $R_1 R_2$, the translation column is $R_1 a_2 + a_1 + v_1 \tau_2$, the velocity column is $R_1 v_2 + v_1$, and the time entry is $\tau_1 + \tau_2$. These are the Galilean composition laws. Injectivity is evident from the entries. $\square$

In this realisation the position of a body is carried by the w -column and its velocity by the t -column. The pure velocity $D_v = G(I, 0, v, 0) = e^{\sum_i v_i N_{i,t}}$ is a nilpotent exponential. An observation at a fixed instant is the projection $(x, w, t) \mapsto x$ restricted to $t = t_0$; two bodies at the same place with different velocities have the same image at t_0 , and their images separate as $x + v(t - t_0)$.

Example 5.2 (a three-joint arm). Consider an arm with a base joint q_0 about the vertical axis, a shoulder q_1 and an elbow q_2 about horizontal axes, and links $l_1 = 0.5$ m, $l_2 = 0.4$ m. Its pose is the product $e^{q_0 M_{xy}} e^{q_1 M_{xz}} e^{l_1 N_{x,w}} e^{q_2 M_{xz}} e^{l_2 N_{x,w}}$, and the hand is its w -column. For $(q_0, q_1, q_2) = (30^\circ, 20^\circ, 50^\circ)$ the hand is $(0.52538, 0.30333, 0.54689)$. Joint rates $(0, 10, -10)$ and $(0, 90, -90)$ degrees per second give hand speeds 0.08727 and 0.78540 m/s (the latter is $\pi/4$), yet both configurations have the same image at t_0 . After 0.2 s the first-order images are $(0.52021, 0.30034, 0.56329)$ and $(0.47885, 0.27646, 0.69449)$.

Proposition 5.3 (the two elbows). *For the planar two-link chain with $l_1 > l_2 > 0$, the map $(q_1, q_2) \mapsto \text{hand}$ is two-to-one away from $q_2 \in \{0, \pi\}$: the configuration (q_1, q_2) and*

$$\left(q_1 + 2 \arctan \frac{l_2 \sin q_2}{l_1 + l_2 \cos q_2}, -q_2 \right)$$

have the same image.

Proof. The hand lies at distance $\rho = |l_1 + l_2 e^{iq_2}|$ from the shoulder, in the direction $q_1 + \arg(l_1 + l_2 e^{iq_2})$. Replacing q_2 by $-q_2$ preserves ρ and negates the argument, which lies in $(-\pi/2, \pi/2)$ because $l_1 > l_2$; the stated change of q_1 restores the direction. By the law of cosines no other configuration has the same ρ and direction, except when $\sin q_2 = 0$. $\square$

For the example above the second configuration has shoulder 64.06809° and elbow -50° ; its hand agrees with the first to all printed digits.

5.3 The relativistic form

In units with $c = 1$ the Lorentz boost along x is $e^{\varphi K_{x,t}}$ by Theorem 2.3, with rapidity $\varphi = \text{artanh } v$. For $\varphi = 0.5$ its (x, t) block has diagonal $\cosh 0.5 = 1.12763$ and off-diagonal $\sinh 0.5 = 0.52110$. For the faster arm above, $v = 0.78540$ metres per second, the rapidity is 2.6×10^{-9} , and the Galilean shear of Proposition 5.1 is its contraction.

6. RANK, SEPARABILITY AND A SINGLE OBJECT

6.1 Separability as a rank condition

A state of a composite system with component spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ is an element of $\mathbb{R}^m \otimes \mathbb{R}^n \cong M_{m \times n}(\mathbb{R})$. It is a *product*, describable as a pair of independent states, if it has the form $x \otimes y = xy^\top$.

Proposition 6.1. $\psi \in M_{m \times n}(\mathbb{R})$ is a product if and only if $\text{rank } \psi \leq 1$. More generally, by the singular value decomposition $\psi = \sum_k \sigma_k u_k v_k^\top$ (Schmidt 1907), the least number of products whose sum is ψ equals $\text{rank } \psi$, the number of non-zero σ_k .

The real matrix $\psi = (e_0 e_1^\top - e_1 e_0^\top) / \sqrt{2}$ has singular values $(1/\sqrt{2}, 1/\sqrt{2})$ and rank two. It is not a pair of states. The description of such a system as “two objects” is the hypothesis of rank one, and it is false for ψ .

6.2 A twelve-component model

We now give a finite real model of one object observed in two places. Let the state space be $\mathbb{R}^{12}$ with coordinates

$$(x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2, o, e_1, e_2, r),$$

where the first two groups are the place and time at which the object is observed as A and as B , o records whether it has been observed, e_1, e_2 are the outcomes displayed by the two views, and $r \in \{\pm 1\}$ is a further component. The two views are the rank-five coordinate projections

$$P_A = \sum_{k \in \{x_1, y_1, z_1, t_1, e_1\}} E_{kk}, \quad P_B = \sum_{k \in \{x_2, y_2, z_2, t_2, e_2\}} E_{kk}.$$

Separation of the views is the shear $S_u = I - uN_{x_1, t_1} + uN_{x_2, t_2}$; observation sets $o = 1$; and the coupling is the nilpotent matrix

$$N = N_{e_1, r} - N_{e_2, r}, \quad N^2 = 0, \quad C(o) = e^{oN} = I + oN.$$

Proposition 6.2. For every state s with $e_1 = e_2 = 0$ and $o = 1$, the state $C(1)s$ satisfies $e_1 = r$ and $e_2 = -r$; in particular $e_1 + e_2 = 0$. The shear S_u changes only x_1 and x_2 , and commutes with $C(o)$. For $o = 0$, $C(0) = I$.

Proof. $Ns = r(e_{e_1} - e_{e_2})$, which gives the first claim. $S_u - I$ and N involve disjoint sets of rows and columns, so they commute, and S_u acts only on the coordinates x_1, x_2 . $\square$

Starting from $(0, 0, 0, 5, 0, 0, 0, 5, 0, 0, 0, 1)$ and separating with $u = 1000$, the views are 5000 units apart along x ; after observation and coupling, P_A displays $e_1 = 1$ and P_B displays $e_2 = -1$. A single matrix acting on a single vector fixed both displayed outcomes in one step. In the coordinates of either view taken alone, the change of the other is unexplained.

6.3 Correlations and the CHSH bound

Suppose now that each view measures along an angle, a for A and b for B , with outcomes in $\{\pm 1\}$. A *two-object model* is one in which A 's outcome is a function $A(a, \lambda)$ and B 's a function $B(b, \lambda)$ of the local setting and a shared variable λ with some probability distribution.

Theorem 6.3 (Bell; Clauser, Horne, Shimony and Holt). *In a two-object model the correlation $E(a, b) = \mathbb{E}[A(a, \lambda)B(b, \lambda)]$ satisfies*

$$|E(a, b) - E(a, b') + E(a', b) + E(a', b')| \leq 2.$$

Proof. For fixed λ the quantity $A(a)(B(b) - B(b')) + A(a')(B(b) + B(b'))$ equals ± 2 , since one bracket is 0 and the other ± 2 . Averaging over λ gives the bound (Bell 1964; Clauser, Horne, Shimony and Holt 1969). $\square$

A two-object model with λ uniform on the circle and $A = \text{sgn} \cos(a - \lambda)$, $B = -\text{sgn} \cos(b - \lambda)$ gives $E(a, b) = -(1 - 2|a - b|/\pi)$ for $|a - b| \leq \pi$, and at the angles $a = 0$, $a' = \pi/2$, $b = \pi/4$, $b' = 3\pi/4$ it attains $|S| = 2$. The correlation measured in nature for a spin singlet is $E(a, b) = -\cos(a - b)$, for which $|S| = 2\sqrt{2} = 2.8284$, the largest value quantum theory allows (Tsirelson 1980).

Now extend the twelve-component model by one component μ , uniform on $[0, 1]$ and independent of r , and let the coupling depend on both settings: $e_1 = r$, and $e_2 = -r$ if $\mu < \cos^2 \frac{a-b}{2}$, $e_2 = r$ otherwise. Then

$$E(a, b) = -\left(\cos^2 \frac{a-b}{2} - \sin^2 \frac{a-b}{2}\right) = -\cos(a - b),$$

so this model attains $2\sqrt{2}$. It escapes the theorem precisely because e_2 depends on a : it is not a two-object model. The marginal distribution of e_2 is uniform on $\{\pm 1\}$ whatever a is, so the dependence cannot be used to transmit a message from A to B . In the language of this paper, the model is local in $\mathbb{R}^{12}$, where one matrix acts on one vector, and non-local only in its projections.

7. INFINITELY MANY COMPONENTS

7.1 *A finite length in infinitely many directions*

In ℓ^2 , the vector with coordinates $u_k = 2^{-k/2}$, $k \geq 1$, has infinitely many non-zero components and $\|u\|^2 = \sum_k 2^{-k} = 1$. Its partial sums are $1 - 2^{-t}$; in double precision the partial sum first equals 1 at $t = 54$, because $1 - 2^{-54}$ lies exactly halfway between 1 and its floating-point predecessor and rounds to 1.

7.2 *The tower of rotation groups*

$\mathbb{R}^n$ has $\binom{n}{2}$ coordinate planes, and the generators M_{pq} , $p < q$, span the Lie algebra $\mathfrak{so}(n)$. The inclusions $\text{SO}(n) \subset \text{SO}(n+1)$ have quotients $\text{SO}(n+1)/\text{SO}(n) \cong S^n$, and Theorem 2.3 does not depend on n : the same three terms turn a plane in every dimension. Bott's periodicity theorem shows that the homotopy groups of the direct limit $\text{O} = \bigcup_n \text{O}(n)$ repeat with period eight (Bott 1959).

7.3 *Dimension is not cardinality*

The phrase “more components” requires care. By Cantor's theorem $|\mathbb{R}| = |\mathbb{R}^n|$ for every n (Cantor 1878); by Brouwer's invariance of dimension there is no homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$ (Brouwer 1911). A lift adds structure, not points: the new components are new independent directions of a linear, or at least topological, space.

7.4 A square root of the Laplacian

The most consequential lift in physics is Dirac's. Set $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = iM_{01}$, $Z = \text{diag}(1, -1)$. Given $\gamma_1, \dots, \gamma_n$ with $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}I$, the $n + 2$ matrices $X \otimes \gamma_a$, $Y \otimes I$, $Z \otimes I$ satisfy the same relations (Brauer and Weyl 1935), and for any such family the operator $D = \sum_a \gamma_a \partial_a$ on $C^2(\mathbb{R}^n, \mathbb{C})$ satisfies $D^2 = \Delta$: expanding, $D^2 = \sum_{a,b} \gamma_a \gamma_b \partial_a \partial_b = \frac{1}{2} \sum_{a,b} (\gamma_a \gamma_b + \gamma_b \gamma_a) \partial_a \partial_b = \Delta$ by the symmetry of second derivatives. A second-order scalar equation becomes a first-order equation at the price of $2^{\lfloor n/2 \rfloor}$ components (Dirac 1928; Lawson and Michelsohn 1989).

8. CONCLUDING REMARKS

The results above share one moral, which can be stated without metaphor. The solutions of an equation, the objects of a composite state and the quantities visible in an observation are not properties of the equation, the state or the system alone, but of the pair formed with an ambient space. A quadratic with negative discriminant has no roots, two roots, a hyperboloid or a sphere of roots according to the algebra in which it is read; a rank-two state is one object that no pair of objects can represent; an orthogonal projection turns rotations into annihilations and reflections. When a problem looks impossible, or two things seem to act on each other without contact, it is worth asking whether one is looking at a projection, and what space it is a projection of.

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